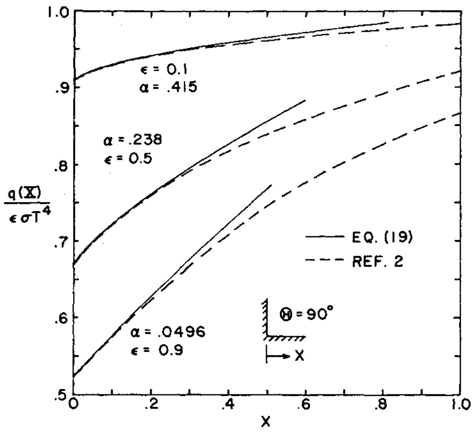

Fig. 3 Asymptotic behavior.

where

$$b \equiv (1 + \cos\theta)\csc^2\theta$$

$$Q(\alpha,\theta) \equiv \int_0^\infty u^{1-\alpha}K(1,u)du,\ 0 < \alpha < 1 \tag{10}$$

A term that varies like X was omitted in Eq. (8) because it would generate a term like $X \ln X$ on the right-hand side of Eq. (9), which is not permissible since $X \ln X$ dominates X as $X \to 0$.

Equating like powers of X gives

$$X^0:\quad \beta_0 = 1 + \lambda b \beta_0 \tag{11}$$

$$X:\quad 0 = \beta_0 + (\beta_1/\alpha) \tag{12}$$

$$X^{1-\alpha}:\quad 1 = \lambda Q(\alpha,\theta) \tag{13}$$

Solving Eq. (11) for β_0 gives $\beta_0 = \beta(0)$ as given by Eq. (6). The value of β_1 is determined from Eq. (12) as $\beta_1 = -\alpha\beta(0)$. Equation (13) determines α as a function of λ and θ.

The explicit inversion of Eq. (13) for α as a function of θ and λ is quite laborious in general. We can, however, obtain an approximate expression for small α which illustrates the behavior. Using Eq. (10), one can show

$$Q(\alpha,\theta) = \frac{1}{\alpha} + \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\mu_n(\theta) \tag{14}$$

where

$$\mu_n(\theta) \equiv \int_1^\infty \frac{(\ln u)^n}{u}[u^2K(1,u) - 1]\,du + \int_0^1 u(\ln u)^n K(1,u)du \tag{15}$$

Using standard integral tables, we evaluate μ_0 as

$$\mu_0 = \ln\left[\frac{2}{1-\cos\theta}\right] + \frac{2\cos\theta - 1}{1 - \cos\theta} \tag{16}$$

For small α it follows that

$$\alpha = \lambda/(1 - \lambda\mu_0) + 0(\alpha^2) \tag{17}$$

When $\theta = \pi/2$, $Q(\alpha,\pi/2)$ can be evaluated in terms of gamma functions. We then obtain

$$\lambda = \alpha\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - \alpha/2)\Gamma(1 + \alpha/2)} \tag{18}$$

For small X, the heat transfer is

$$\frac{q(X)}{\epsilon\sigma T^4} \sim \frac{\sin^2(\theta/2)}{1 - \rho\cos^2(\theta/2)} + \frac{\epsilon\alpha\beta(0)}{\rho}X^{1-\alpha},\ X \to 0 \tag{19}$$

This expression shows the nonanalytic behavior when $X \to 0$. Since $0 < \alpha < 1$, the derivative of $q(X)$ is infinite at the vertex. Unless especially accounted for, this behavior causes difficulty in numerical solutions. The singular behavior becomes stronger as α increases.

Equation (19) is plotted in Fig. 3 for $\theta = \pi/2$, in which case Eq. (18) can be used to evaluate α. Also shown are the numerical results of Love and Turner,[2] which agree quite well with Eq. (19) at the vertex for this case. Moreover, Eq. (19), although valid only for small X, gives good results over a larger range than was strictly intended.

References

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Effect of Implementation Delays on Errors in Linear Estimators

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Nomenclature

H	= measurement matrix
K	= filter gain matrix
$P(-)$	= covariance matrix of estimation errors, prior to an update
$P(+)$	= covariance matrix of estimation errors, after an update
$\mathbf{u}$	= vector of system disturbances (process noise)
$\mathbf{v}$	= vector of measurement noise
$\mathbf{x}$	= state vector, prior to a correction
$\mathbf{x}'$	= state vector, immediately after a correction
$\hat{\mathbf{x}}(-)$	= estimate of state, immediately prior to incorporating a new measurement
$\hat{\mathbf{x}}(+)$	= estimate of state, after incorporating a new measurement in the estimate but prior to correcting the state
$\hat{\mathbf{x}}'$	= estimate of state, immediately after correcting the state
$\tilde{\mathbf{x}}(-)$	= error in the state estimate, prior to incorporating a new measurement
$\tilde{\mathbf{x}}(+)$	= error in the state estimate, immediately after incorporating a measurement
$\mathbf{z}$	= measurement vector
$\Phi(t_i,t_j)$	= state transition matrix relating change in state vector over interval t_j to t_i

Subscripts

t_n	= pertaining to time $t = t_n$
n	= also pertaining to time $t = t_n$

Superscripts

*	= the version implemented in the linear estimator
T	= transpose of a vector or matrix

Two common problems facing the designer of a Kalman estimator are the finite operation times inherent in any digital computer implementation and the modeling of correlations in the measurement errors. It is frequently impractical to provide a complete treatment of correlated measure-

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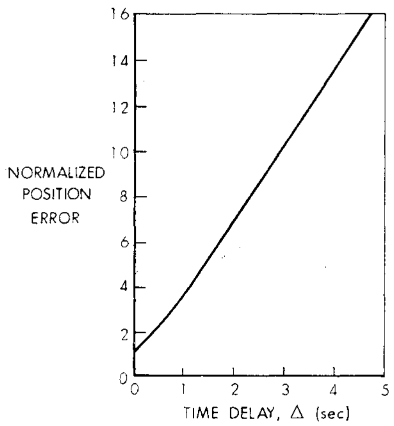

Fig. 1 Effect of calculation delay on the accuracy of an aided inertial navigator.

ment errors because of the additional filter complexity required. The designer often accepts the accuracy penalty imposed by treating measurement errors as a white sequence. Filter implementation delays can also be ignored, by paying an accuracy penalty that varies according to the delay imposed (see Fig. 1). This Note illustrates a way to avoid accuracy degradation caused by implementation delays in the discrete Kalman filter. In particular, the results are shown to be valid for that class of suboptimal filters in which the designer chooses to ignore temporal correlations in the measurement noise. The results have a particular application in analyzing aided inertial navigation systems.

Estimates of the state which are calculated by the filter after each measurement are only applicable at the instants the measurements are made. Because of time consumed in data processing these estimates are not necessarily proper for use in correcting the state at the moment they first become available. In an effort to improve the correction so that it is more accurate at the time the state is updated, the estimates can be propagated over the delay interval Δ, using the state transition matrix;

$$\hat{\mathbf{x}}_{t_n+\Delta}(+) = \Phi^*(t_n + \Delta, t_n)\hat{\mathbf{x}}_{t_n}(+) \tag{1}$$

The following development shows analytically that if the state transition matrix used in the filter is properly related to that of the system, the delay Δ causes no increase in errors when Eq. (1) is used.

In general $\hat{\mathbf{x}}_{t_n}(+)$, the updated estimate of the state vector at time t_n based on the measurement taken at that time, is computed from[1]

$$\hat{\mathbf{x}}_{t_n}(+) = \hat{\mathbf{x}}_{t_n}(-) + K_{t_n}[\mathbf{z}_{t_n} - H_{t_n}{}^*\hat{\mathbf{x}}_{t_n}(-)] \tag{2}$$

Let us define the state being estimated by the filter to be a deviation from a nominal set of variables and assume that the nominal state is corrected whenever a new estimate becomes available. This is usually the format presented for the optimal mixing of navigation data, but any linear filtering problem can be cast in this manner. Due to the mechanization delay, the new estimate becomes available Δ sec after the measurement and the state vector is corrected at $t_n + \Delta$ according to

$$\mathbf{x}'_{t_n+\Delta} = \mathbf{x}_{t_n+\Delta} - A\hat{\mathbf{x}}_{t_n+\Delta}(+) \tag{3}$$

where the matrix A is used to account for any difference between the true state vector and that implemented in the filter. When the relation expressed by Eq. (1) is used

$$\mathbf{x}'_{t_n+\Delta} = \mathbf{x}_{t_n+\Delta} - A\Phi^*(t_n + \Delta, t_n)\hat{\mathbf{x}}_{t_n}(+) \tag{4}$$

As can be seen from Eq. (3), after the update is made the corrected filter state is equal to the difference between the state before correction and the estimate of that state. This difference is the error in the estimate $[(\mathbf{x}_{t_n}{}'_{+\Delta} = \tilde{\mathbf{x}}_{t_n+\Delta}(+)]$. Since the correction results in subtracting the estimate from the filter state, the estimate of the corrected state is zero $[\hat{\mathbf{x}}_{t_n+\Delta}{}' = 0]$. Also, because no new information is obtained by the filter between measurements.

$$\hat{\mathbf{x}}_{t_{n+1}}(-) = \mathbf{0}$$
$$\mathbf{x}_{t_{n+1}} = \tilde{\mathbf{x}}_{t_{n+1}}(-) \tag{5}$$

That is, the filter estimate prior to each new measurement is zero and the deviation from nominal at that time is identical with the estimation error. Therefore, Eq. (2) reduces to

$$\hat{\mathbf{x}}_{t_n}(+) = K_{t_n}\mathbf{z}_{t_n} = K_{t_n}[H_{t_n}\mathbf{x}_{t_n} + \mathbf{v}_{t_n}] \tag{6}$$

Substituting Eq. (6) into Eq. (4) it follows that

$$\mathbf{x}'_{t_n+\Delta} = \mathbf{x}_{t_n+\Delta} - A\Phi^*(t_n + \Delta, t_n)K_{t_n}[H_{t_n}\mathbf{x}_{t_n} + \mathbf{v}_{t_n}] \tag{7}$$

But (simplifying the subscripted time notation),

$$\mathbf{x}_{n+\Delta} = \Phi(t_n + \Delta, t_n)\mathbf{x}_n + \mathbf{w}_n$$
$$\mathbf{w}_n \triangleq \int_{t_n}^{t_n+\Delta} \Phi(t_n + \Delta, \tau)\mathbf{u}(\tau)d\tau \tag{8}$$

where $\mathbf{u}(t)$ is a vector of white process noise causing deviations of the state from its nominal value. Therefore, Eq. (7) can be written,

$$\mathbf{x}'_{n+\Delta} = [\Phi(t_n + \Delta, t_n) - A\Phi^*(t_n + \Delta, t_n)K_nH_n]\mathbf{x}_n + \mathbf{w}_n - A\Phi^*(t_n + \Delta, t_n)K_n\mathbf{v}_n \tag{9}$$

The estimation error covariance after the state is corrected at $t_n + \Delta$ is

$$P_{n+\Delta}(+) = \langle\tilde{\mathbf{x}}_{n+\Delta}(+)\tilde{\mathbf{x}}_{n+\Delta}{}^T(+)\rangle = \langle\mathbf{x}'_{n+\Delta}\mathbf{x}'_{n+\Delta}{}^T\rangle = [\Phi(t_n + \Delta, t_n) - A\Phi^*(t_n + \Delta, t_n)K_nH_n]P_n(-) \times [\Phi(t_n + \Delta, t_n) - A\Phi^*(t_n + \Delta, t_n)K_nH_n]^T + W_n + A\Phi^*(t_n + \Delta, t_n)K_nR_nK_n{}^T\Phi^{*T}(t_n + \Delta, t_n)A^T \tag{10}$$

where

$$P_n(-) \triangleq \langle\tilde{\mathbf{x}}_n(-)\tilde{\mathbf{x}}_n{}^T(-)\rangle = \langle\mathbf{x}_n\mathbf{x}_n{}^T\rangle$$
$$R_n \triangleq \langle\mathbf{v}_n\mathbf{v}_n{}^T\rangle$$
$$W_n \triangleq \langle\mathbf{w}_n\mathbf{w}_n{}^T\rangle$$

The estimation error covariance prior to a measurement at $t_{n+1}, P_{n+1}(-)$, is given by

$$P_{n+1}(-) = \Phi(t_{n+1}, t_n + \Delta)P_{n+\Delta}(+)\Phi^T(t_{n+1}, t_n + \Delta) + Y_n \tag{11}$$

where

$$Y_n = \langle\mathbf{y}_n\mathbf{y}_n{}^T\rangle; \quad \mathbf{y}_n = \int_{t_n+\Delta}^{t_{n+1}} \Phi(t_{n+1}, \tau)\mathbf{u}(\tau)d\tau$$

Thus $\mathbf{y}_n$ is the effect of noise on the state over the interval $t_n + \Delta$ to t_{n+1}. If $A\Phi^* = \Phi A$, combining Eqs. (10) and (11)

$$P_{n+1}(-) = \Phi(t_{n+1}, t_n)[I\text{-}AK_nH_n]P_n(-)[I\text{-}AK_nH_n]^T \times \Phi^T(t_{n+1}, t_n) + \Phi(t_{n+1}, t_n + \Delta)W_n\Phi^T(t_{n+1}, t_n + \Delta) + \Phi(t_{n+1}, t_n)AK_nR_nK_n{}^TA^T\Phi^T(t_{n+1}, t_n) + Y_n \tag{12}$$

Expanding the terms containing Y_n and W_n,

$$\Phi(t_{n+1}, t_n + \Delta)W_n\Phi^T(t_{n+1}, t_n + \Delta) + Y_n = \Phi(t_{n+1}, t_n + \Delta)\langle\mathbf{w}_n\mathbf{w}_n{}^T\rangle\Phi^T(t_{n+1}, t_n + \Delta) + \langle\mathbf{y}_n\mathbf{y}_n{}^T\rangle = \Phi(t_{n+1}, t_n + \Delta)\left\langle\left(\int_{t_n}^{t_n+\Delta}\Phi(t_n + \Delta, \tau)\mathbf{u}(\tau)d\tau\right) \times \left(\int_{t_n}^{t_n+\Delta}\Phi(t_n + \Delta, \tau)\mathbf{u}(\tau)d\tau\right)^T\right\rangle\Phi^T(t_{n+1}, t_n + \Delta) + \left\langle\left(\int_{t_n+\Delta}^{t_{n+1}}\Phi(t_{n+1}, \tau)\mathbf{u}(\tau)d\tau\right)\left(\int_{t_n+\Delta}^{t_{n+1}}\Phi(t_{n+1}, \tau)\mathbf{u}(\tau)d\tau\right)^T\right\rangle \tag{13}$$

Since $\Phi(t_{n+1},t_n + \Delta)$ is deterministic, it may be placed inside of the ensemble average operator and the integration sign. Also, since $\mathbf{w}_n$ and $\mathbf{y}_n$ are integrals of white noise over separate intervals their cross-correlation must be zero;

$$\langle \mathbf{y}_n \mathbf{w}_n^T \rangle = \langle \mathbf{w}_n \mathbf{y}_n^T \rangle = 0 \tag{14}$$

and

$$\langle \mathbf{y}_n \mathbf{y}_n^T \rangle + \langle \mathbf{w}_n \mathbf{w}_n^T \rangle = \langle (\mathbf{w}_n + \mathbf{y}_n)(\mathbf{w}_n + \mathbf{y}_n)^T \rangle \tag{15}$$

Using Eq. (15), and the fact that

$$\Phi(t_{n+1},t_n + \Delta)\Phi(t_n + \Delta,\tau) = \Phi(t_{n+1},\tau) \tag{16}$$

Eq. (13) becomes

$$\Phi(t_{n+1},t_n + \Delta)W_n\Phi^T(t_{n+1},t_n + \Delta) + Y_n = \left\langle \left(\int_{t_n}^{t_{n+1}} \Phi(t_{n+1},\tau)\mathbf{u}(\tau)d\tau\right)\left(\int_{t_n}^{t_{n+1}} \Phi(t_{n+1},\tau)\mathbf{u}(\tau)d\tau\right)^T \right\rangle \triangleq Q_n \tag{17}$$

and Eq. (12) becomes

$$P_{n+1}(-) = \Phi(t_{n+1},t_n)\{[I\text{-}AK_nH_n]P_n(-)[I\text{-}AK_nH_n]^T + AK_nR_nK_n^TA^T\}\Phi^T(t_{n+1},t_n) + Q_n \tag{18}$$

It can be seen that the preceding relation for the error covariance, Eq. (18), is identical to the one obtained when there is no time delay. Therefore, if $A\Phi^* = \Phi A$ and Eq. (1) is employed, the time delay has no effect on errors subsequent to the last update and correction, Δ sec after the final measurement. There is an unavoidable difference in errors over the interval between a measurement and the correction which follows it. The condition on the matrices A and Φ^* is satisfied if all states of the problem except those representing correlated measurement errors are corrected after each measurement.

Reference

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Finite Deflection of Elliptical Plates on Elastic Foundations

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Nomenclature

D	= flexural rigidity of plate
E	= modulus of elasticity of plate material
G	= shear modulus
K	= dimensionless foundation modulus
Q	= dimensionless uniformly distributed load
R	= aspect ratio, b/a
U,V,W	= dimensionless displacement parameters
W_0	= dimensionless lateral center deflection, w_0/h
$2a,2b$	= length of minor and major axis of elliptical plate, respectively
h	= thickness of plate
k	= elastic foundation reaction per unit area per unit deflection
q	= intensity of uniformly distributed load
u,v,w	= displacement along the x, y, and z axes, respectively

THE finite deflection behavior of thin elastic plates was first formulated mathematically by von Kármán in the form of two coupled nonlinear partial differential equations.[1] Approximate solutions for these equations have been obtained for the case of the rectangular plate by Way[2] using the Rayleigh-Ritz technique, and also by Levy and Greenman,[3] who substituted a double Fourier series solution into the differential equations and evaluated the coefficients. Invariably, the methods of analysis employed are extremely laborious and require considerable computations.

In this work, a simple and yet sufficiently accurate method based on the small parameter perturbation technique is used to analyze the small and finite deflection behavior of uniformly loaded clamped elliptical plates resting on an elastic foundation. This technique has been applied successfully to circular plates with no elastic support by Chien,[4] and, more recently, to a square plate by Chien and Yeh.[5]

Throughout the following analysis, the plate is considered to be elastic and the foundation is considered to be of the Winkler type; i.e., foundation reaction is proportional to the deflection.

For plates resting on elastic foundation and with moderately large deflections, the von Kármán equations can be slightly modified and three nonlinear partial differential equations governing the lateral deflection and in plane displacements can be written as

$$u_{,xx} + w_{,x}w_{,xx} + \nu(v_{,xy} + w_{,y}w_{,xy}) + \tfrac{1}{2}(1-\nu)(u_{,yy} + v_{,xy} + w_{,x}w_{,yy} + w_{,y}w_{,xy}) = 0 \tag{1}$$

$$v_{,yy} + w_{,y}w_{,yy} + \nu(u_{,xy} + w_{,x}w_{,xy}) + \tfrac{1}{2}(1-\nu)(v_{,xx} + u_{,xy} + w_{,y}w_{,xx} + w_{,x}w_{,xy}) = 0 \tag{2}$$

$$D\nabla^2\nabla^2 w = q - kw + h\left\{\frac{E}{(1-\nu^2)}\left[u_{,x} + \frac{1}{2}(w_{,x})^2 + \nu\left(v_{,y} + \frac{1}{2}(w_{,y})^2\right)\right]w_{,xx} + \frac{E}{(1-\nu^2)}\left[v_{,y} + \frac{1}{2}(w_{,y})^2 + \nu\left(u_{,x} + \frac{1}{2}(w_{,x})^2\right)\right]w_{,yy} + \frac{E}{(1-\nu^2)} \times (u_{,y} + v_{,x} + w_{,x}w_{,y})w_{,xy}\right\} \tag{3}$$

where ∇ is the Laplacian operator, ν being Poisson's ratio, D the flexural rigidity, E the modulus of elasticity of the plate material; q the intensity of uniformly distributed load, k the foundation modulus; h is the thickness of the plate and the comma notation signifies differentiation.

By adopting the dimensionless ratios

$$R = b/a;\ \xi = x/a;\ \eta = y/b$$

$$U = \frac{ua}{h^2};\ V = \frac{va}{h^2};\ W = \frac{w}{h};\ Q = \frac{qb^4}{Dh};\ K = \frac{kb^4}{D}$$

Table 1 Coefficients w for maximum small deflection at center for various elastic foundation moduli plate aspect ratios $R = b/a$; $W_{\max} = w(b^4q/D)(10^{-2})$

Dimensionless Foundation modulus K	R = 1	R = 1.25	R = 1.50	R = 1.75	R = 2.0
0	1.5625	0.9294	0.5510	0.3355	0.2118
20	1.3017	0.8300	0.5140	0.3212	0.2060
40	1.1143	0.7495	0.4816	0.3080	0.2004
60	0.9730	0.6998	0.4529	0.2958	0.1951
80	0.8627	0.6268	0.4273	0.2846	0.1900
100	0.7741	0.5791	0.4044	0.2741	0.1852
120	0.7013	0.5378	0.3838	0.2643	0.1807
140	0.6405	0.5019	0.3651	0.2552	0.1763
160	0.5889	0.4703	0.3480	0.2467	0.1721
180	0.5446	0.4422	0.3324	0.2387	0.1682
200	0.5060	0.4172	0.3181	0.2312	0.1644

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